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Serre duality for rigid analytic spaces

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ABSTRACT

In this paper a duality theory for morphism $X \to Y$ of rigid analytic spaces over some non-archimedean valued complete field K is developed. In order to keep the statements and the proofs reasonable we make the restrictions Y is affinoid and $X \to Y$ is smooth and proper.

The successful approach of B. Chiarellotto [Ch] for Stein spaces is the basis for this paper. An essential step is to show a strong form of unicity for the residue maps Res. One uses further a suitable definition for cohomology with compact support; the work of R. Kiehl [Kl] on proper mappings; geometric points as introduced in [P]; the strong G-topology of [K1, K2] in the version of [BGR].

As one expects the paper is rather technical in nature and the results are not surprising. However for the investigation of proper rigid analytic manifolds the Serre-duality can not be missed. We thank B. Chiarellotto for his careful reading of the manuscript, which has led to improvements of the paper.

1. COHOMOLOGY WITH COMPACT SUPPORT

In this section $p: X \to Y$ is a morphism of analytic varieties over some complete non-archimedean valued base field K. We assume that X is separated and that Y is affinoid. On X and Y we use the strong G-topology as explained in [BGR] pp. 336–346.

1.1. LEMMA. *Let affinoid subsets $Z_1, \dots, Z_r, Z_1', \dots, Z_r'$ of X be given such that $Z_i \Subset_Y Z_i'$ for all i. Put $Z = Z_1 \cup \dots \cup Z_r$ and $Z' = Z_1' \cup \dots \cup Z_r'$. Then $\{X - Z, Z'\}$ is an admissible covering of X w.r.t. the strong G-topology.*

PROOF. Let $\{X_i\}_{i\in I}$ be an admissible affinoid covering of X. Then $\{X-Z, Z'\}$ is admissible if for every $i\in I$ also $\{X_i/Z\cap X_i,\ Z'\cap Z_i'\}$ is admissible. We note that $X_i\cap Z_j \Subset_{X_i\times Y} X_i\cap Z_j'$ holds for $i\in I$ and $1\le j\le r$. The map $X_i\cap Z_j'\to X_i\times Y$ is given by $X_i\cap Z_j'\subset X_i\times Z_j' \xrightarrow{(1,p)} X_i\times Y$. Thus we may suppose that X is also an affinoid space. The covering $\{Z_1,\dots,Z_r\}$ of Z can be replaced by a covering of Z by rational subspaces of X. Hence we may suppose that each Z_j is already a rational subspace of X. If we have shown that each $\{X-Z_j, Z_j'\}$ is admissible then also each $\{X-Z_j, Z'\}$ is admissible. Since $\{X-Z, Z'\}$ has as refinement $\bigcap_{j=1,\dots,r}\{X-Z_j, Z'\}$ also $\{X-Z, Z'\}$ is admissible.

Thus we are reduced to the case $r=1$. Let $Z\subset X$ be given by inequalities:

$$|f_0|\ge |f_1|,\dots,|f_m|.$$

where $\{f_0,\dots,f_m\}$ generate the unit ideal of $O(X)$. For a suitable $\varrho\in|K^*|$ one defines $X_1=\{x\in X \mid |f_0(x)|\ge\varrho\}$ and $X_2=\{x\in X\mid |f_0(x)|\le\varrho\}$. Further $Z\subset X_1$ and $Z\cap X_2=\emptyset$. After replacing X by X_1 and $f_1,\dots,f_m$ by $f_1f_0^{-1},\dots,f_mf_0^{-1}$ we are in the situation that Z is a Weierstrass domain in X given by the inequalities $|f_1|\le 1,\dots,|f_m|\le 1$. Similarly, we may suppose that Z' is a Weierstrass domain in X. Let t_j denote topological generators of $O(Z')$ over $O(Y)$ such that $Z\subset\{z'\in Z' \mid \text{all } |t_j(z')|<1\}$. Since $O(X)$ is dense in $O(Z')$ we can change the elements t_j such that the new t_j belong to $O(X)$ and are topological generators of $O(Z')$ over $O(X)$. It follows that $Z'=\{x\in X \mid \text{all } |t_j|\le 1\}$ and again $Z\subset\{x\in X\mid \text{all } |t_j(x)|<1\}$.

On the affinoid set $A=\{x\in X \mid |t_j|<1\}$ one has $\delta(a):=\max(|f_1(a)|,\dots,|f_m(a)|)>1$ for all $a\in A$. It follows that for some $\varrho_j\in\sqrt{|K^*|}$, $\varrho_j<1$, one has $\delta(a)>\varrho_j^{-1}$ for all $a\in A$. This means that $\{x\in X\mid \max|f_i(x)|\le\varrho^{-1}\}\subset Z'$ for $\varrho=\max\varrho_j$. From [BGR] Prop. 5, p. 343, it follows at once that $X-Z$ is admissible. The covering $\{X-Z, Z'\}$ is admissible since it has the following refinement:

$$\{x\in X\mid |f_1(x)|\ge\varrho^{-1}\},\dots,\{x\in X\mid |f_m(x)|\ge\varrho^{-1}\}, Z'.$$

1.2. DEFINITIONS. Let X be a separated analytic space and $Z\subset X$ a finite union of affinoid subsets. For any abelian sheaf F on X we define

$$H_Z^0(X,F)=\ker(H^0(X,F)\to H^0(X-Z,F)).$$

This makes sense since $X-Z$ is admissible according to (1.1). It is easily seen that $F\mapsto H_Z^0(X,F)$ is a left exact functor. Its derivatives are denoted by $H_Z^i(X,F)$. We remark that the notion of support of a section $s\in H^0(X,F)$ is not so obvious, since one has to use the geometric points of X (see [P]).

1.3. LEMMA. *X, Z and F as above. There exists a long exact sequence*:

$$\begin{aligned}0&\to H_Z^0(X,F)\to H^0(X,F)\to H^0(X-Z,F)\\ &\to H_Z^1(X,F)\to H^1(X,F)\to H^1(X-Z,F)\to\cdots.\end{aligned}$$

PROOF. There exists a resolution $0\to F\to I_0\to I_1\to\cdots$ of F by injective sheaves, such that $H^0(X, I_n) \to H^0(X-Z, I_n)$ is surjective for all $n \geq 0$. Indeed, using the notion of geometric point of X one defines $I_0 = I_0(F)$ as follows:

$$I_0(F)(U) = \prod_{p \ni U} \tilde{F}_p,$$

where the product is taken over all geometric points p of X such that $U \in p$ and where $\tilde{F}_p$ denotes an injective hull of the abelian group $F_p = \varinjlim_{V \in p} F(V)$.

Further $I_1 = I_0(I_0/F)$ etc. One finds an exact sequence of complexes

$$0 \to \bigoplus_{i \geq 0} H_Z^0(X, I_i) \to \bigoplus_{i \geq 0} H^0(X, I_i) \to \bigoplus_{i \geq 0} H^0(X-Z, I_i) \to 0.$$

The resulting cohomology sequence is the one of the lemma.

1.4. DEFINITIONS. By $c = c(X/Y)$ we denote the family of subsets of X of the form $Z_1 \cup \cdots \cup Z_r$ where $Z_1, \ldots, Z_r$ are affinoids such that there exists affinoids $Z_1', \ldots, Z_r'$ with $Z_i \Subset_Y Z_i'$ for all i. Of course c depends on X and $X \to Y$. As usual one defines $H_c^\bullet(X, F) = \varinjlim_{Z \in c} H_Z^\bullet(X, F)$. We call this *cohomology with* (*relative*) *compact support*. We note that for a Stein space X and $Y = a$ point, our definition coincides with the one of [Ch]. Indeed for every affinoid $Z \subset X$ there exists an affinoid $Z' \subset X$ with $Z \Subset Z'$.

Further, for a proper morphism $X \to Y$ one has $H_c^\bullet(X, F) = H^\bullet(X, F)$ since $X = Z_1 \cup \cdots \cup Z_r$ and $Z_i \Subset_Y Z_i'$ holds for suitable affinoids Z_i, Z_i'.

1.5. LEMMA. *For U, an admissible open subspace of X, one has a canonical map $H_{c(U/Y)}^\bullet(U, F) \to H_{c(X/Y)}^\bullet(X, F)$.*

PROOF. Let $Z = Z_1 \cup \cdots \cup Z_r$ and $Z' = Z_1' \cup \cdots \cup Z_r'$ be given with $Z_1, \ldots, Z_r$, $Z_1', \ldots, Z_r'$ affinoids in U and $Z_i \Subset_Y Z_i'$ for all i. Let f belong to $H_Z^0(U, F)$. Define $g \in H_Z^0(X, F)$ by $g|_{Z'} = f|_{Z'}$ and $g|_{X-Z'} = 0$. We note that $Z \in c(X/Y)$ and that the map $f \mapsto g$ yields an isomorphism $H_Z^0(U, F) \to H_Z^0(X, F)$. So we also find isomorphisms $H_Z^\bullet(U, F) \to H_Z^\bullet(X, F)$ for any $\bullet$ and any $Z \in c(U/Y)$. In the limit one obtains morphisms $H_{c(U/Y)}^\bullet(U, F) \to H_{c(X/Y)}^\bullet(X, F)$.

1.6. *The canonical topology on cohomology groups*

Let Z be an affinoid space with corresponding Banach algebra of functions $O(Z)$. Any finitely generated $O(Z)$-module M has a unique topology as Banach $O(Z)$-module. Any $O(Y)$-submodule N of M is closed in M and M induces the topology on N. Any $O(Y)$-linear map between finitely generated $O(Y)$-modules is continuous.

An arbitrary $O(Y)$-module M can be written as the direct limit of its finitely generated submodules. Then M is given the (locally convex) direct limit topology and so M is the strict limit of Banach spaces. Any $O(Y)$-linear map between $O(Y)$-modules is continuous w.r.t. the topologies defined above.

Let F be any coherent sheaf on some separated analytic space X of countable type. Let $\{X_i\}_{i \in I}$ be an admissible countable (or finite) affinoid covering of X.

Then $F(X)=\ker(\prod F(X_i)\to\prod_{i<j}F(X_{ij}))$ is a closed subspace of the Fréchet space $\prod F(X_i)$. Hence $F(X)$ has the structure of Fréchet space and this structure does not depend on the choice of the covering $\{X_i\}$.

Let again $\{X_i\}_{i\in I}$ be an admissible covering of X such that the Čech-complex of F w.r.t. $\{X_i\}_{i\in I}$ calculates the cohomology groups $H^\cdot(X,F)$. Of course one could take an affinoid covering of X but also a covering of X by (quasi-)Stein domains X_i (see [K2]) will do. The corresponding locally convex topology on the complex induces a locally convex topology on the groups $H^\cdot(X,F)$. Using the theorem of Banach applied to Fréchet-spaces one shows that the locally convex topology on $H^\cdot(X,F)$ is unique. Indeed the argumentation in [B–S] lemme 1.32, p. 296, is also valid in our situation. Let X be a (quasi-)Stein space then $H^i(X,F)=0$ for $i\neq 0$. Using (1.3) one defines topologies on $H_Z^\cdot(X,F)$. Further $H_c^\cdot(X,F)=\varinjlim_{Z\in c}H_Z^\cdot(X,F)$ is given the direct limit topology. This topology is defined as the strongest locally convex topology such that all maps $H_Z^\cdot(X,F)\to H_c^\cdot(X,F)$ are continuous.

Let X be again a (quasi-)Stein space and let $X_1\subset X_2\subset X_3\subset\cdots$ be an admissible affinoid covering of X such that $O(X_{n+1})\to O(X_n)$ has a dense image for every $n\geq 1$. One would like to define locally convex topologies on $H^\cdot(F)$ and $H_c^\cdot(F)$ for any quasi-coherent sheaf F. The projective limit topology on $H^0(F)=\varinjlim F(X_n)$ can be seen to be independent of the choice of the covering X_n. However quasi-coherent sheaves on a (quasi-)Stein space do not behave well.

We will illustrate this with the following example. Let $X=\mathbb{A}_K^1=\bigcup X_n$ where $X_n=\{z\in X\mid |z|\leq|\pi|^{-n}\}$ with $\pi\in K^*$, $0<|\pi|<1$, fixed. The quasi-coherent sheaf F is defined by:

(1) $F(X_n)$ is a free $O(X_n)$-module on countable many generators $e_n(i)_{i\geq 1}$.

(2) $\varphi_n: F(X_{n+1})\otimes O(X_n)\to F(X_n)$ is given by $\varphi_n(e_{n+1}(i)\otimes 1)=e_n(i)$ for $i\geq n+2$. For $1\leq i\leq n+1$ one has $\varphi_n(e_{n+1}(i)\otimes 1)=\sum_{j=1}^{n+1}\alpha(i,j)e_n(j)$ where $(\alpha(i,j))\in \mathrm{Gl}(n+1,O(X_n))$ and where $\alpha(1,n+1),\dots,\alpha(n+1,n+1)\in O(X_n)$ are linearly independent over $O(X_{n+1})$.

It follows that $\mathrm{im}(F(X_{n+1})\to F(X_n))\cap(O(X_n)e_n(1)+\dots O(X_n)e_n(n))=0$. From this, one easily obtains $F(\mathbb{A}_K^1)=0$ and with some calculation one finds that $H^1(\mathbb{A}_K^1,F)\neq 0$. A similar reasoning proves that the quasi-coherent sheaf $O_X^{(\mathbb{N})}$ on $X=\mathbb{A}_K^1$ satisfies $H^1(X,O_X^{(\mathbb{N})})\neq 0$. In the sequel we will avoid quasi-coherent sheaves as much as possible.

2. DUALITY FOR $\mathbb{A}_Y^n$

2.1. *Stein spaces*

Let Y denote an affinoid space over K. A (relative) Stein space over Y is a morphism $X\xrightarrow{p}Y$ such that X is separated and has an admissible covering by affinoid subsets $\{U_n\}_{n\in\mathbb{N}}$ satisfying:

For every n there are topological generators $h_1(n),\dots,h_{r(n)}(n)$ of $O(U_n)$ over

$O(Y)$ and some constants $a_n \in \sqrt{|K^*|}$ with $0<a_n<1$ such that

$$U_{n-1} = \{u \in U_n \mid |h_i(n)(u)| \leq a_n \quad \text{for } i=1,\dots,r(n)\}.$$

According to [K2], X is a quasi-Stein space. We assume now that $r=:\sup\{r(n) \mid n \in \mathbb{N}\}$ is finite. One can show that X has a closed immersion into some $\mathbb{A}_Y^N := \mathbb{A}_K^N \times Y$. Conversely any closed analytic subspace of $\mathbb{A}_Y^N$ is a Stein space over Y with $r<\infty$.

For absolute Stein spaces the existence of a closed immersion into some $\mathbb{A}_K^N$ has been proved in [L1]. This proof can be copied almost verbatim for the relative case. We indicate the few changes that one has to make.

The condition $r<\infty$ yields that $\Omega_{X/Y,x}$ is generated by $\leq r$ elements for every $x \in X$. As in [L1] §4, p. 33, it follows that $\Omega_{X/Y}$ is globally generated by finitely many sections $d(f)$ with $f \in O(X)$. In the definition of $V(f_1,\dots,f_r)$ on p. 35, one has to replace $X \times X$ by the fibre product of X with itself over Y. The method yields the existence of many $(f_1,\dots,f_s) \in O(X)^s$ such that the map $X \to \mathbb{A}_Y^s$, $x \mapsto (f_1(x),\dots,f_s(x),p(x))$, is injective. In lemma 4.14 of [L1] one has to replace n by r. For every i, there are elements $h_i(i),,\dots,h_r(i)$ such that $O(Y)\langle h_1(i),\dots,h_r(i)\rangle \to O(U_i)$ is surjective. The conclusion of [L1] lemma 4.14, is the existence of many $(f_1,\dots,f_r) \in O(X)^r$ such that for every i, the map

$$O(Y)\langle b_{1,i}f_1,\dots,b_{r,i}f_r\rangle \to O(U_i)$$

is finite. A combination of the three ingredients above leads to the existence of the closed immersion $X \to \mathbb{A}_Y^N$ for any field K.

We note that the embedding theorem for relative Stein spaces over Y is used in (5.3). The spaces U_i etc, in (5.3) are closed subspaces of a product of Y with an open polydisc B. A closed immersion of B in some $\mathbb{A}_K^N$ induces a closed immersion of U_i in some $\mathbb{A}_Y^N$.

2.2. *The Yoneda-pairing*

Let Y be affinoid, X separated and $X \to Y$ a morphism. Let c denote $c(X/Y)$. The category of O_X-modules has enough injective elements. Indeed let F denote some O_X-module. For any geometric point p of X one chooses an injective hull F_p^+ of the stalk F_p seen as $(O_X)_p$-module. The sheaf I defined by

$$I(U) = \prod_{p \ni U} F_p^+$$

is an injective O_X-module and the natural morphism $F \to I$ is injective.

For three O_X-modules A, B, C one can define, using injective resolutions, the Yoneda-pairings

$$\operatorname{Ext}^p(A,B) \times \operatorname{Ext}_c^q(B,C) \to \operatorname{Ext}_c^{p+q}(A,C)$$
$$\operatorname{Ext}_c^p(A,B) \times \operatorname{Ext}^q(B,C) \to \operatorname{Ext}_c^{p+q}(A,C).$$

Here Ext_c^p denotes the derived functors of Ext_c^0, where $\operatorname{Ext}_c^0(F,G) = H_c^0(X, \mathbf{Hom}(F,G))$.

2.3. *The theorems*

Let $X \to Y$ be smooth, Y affinoid and X separated. Let n be the relative dimension of X. Put $\omega = \omega_{X/Y} = \Lambda^n \Omega^1_{X/Y}$ = the sheaf of holomorphic n-forms on X/Y. Let c denote $c(X/Y)$.

For $X \to Y$ smooth and either Stein or proper, we will define a residue map $\mathrm{Res}_X : H^n_c(\omega) \to O(Y)$. Let M be any finitely generated $O(Y)$-module. We will show that $H^n_c(\omega \otimes M) \cong H^n_c(\omega) \otimes M$. The trace map $\mathrm{Tr}_{X,M} : H^n_c(\omega \otimes M) \to M$ is given as $\mathrm{Res}_X \otimes 1_M$.

Res_X and $\mathrm{Tr}_{X,M}$ are continuous and $O(Y)$-linear. In the Yoneda pairings we substitute $A = O_X$, $B = F$ = any coherent sheaf on X, $C = \omega \otimes M$ with M any $O(Y)$-module. We observe that $\mathrm{Ext}^p_c(O_X, -) = H^p_c(-)$ and $\mathrm{Ext}^p(O_X, -) = H^p(-)$. Using the Yoneda pairings we find two morphisms:

(a) $\quad \mathrm{Ext}^{n-i}_c(F, \omega \otimes M) \to \mathrm{Hom}_{O(Y)}(H^i(F), M)$

(b) $\quad \mathrm{Ext}^{n-i}(F, \omega \otimes M) \to \mathrm{Hom}_{O(Y)}(H^i_c(F), M)$.

The duality theorems are:

For $X \to Y$ a (relative smooth) Stein space we will prove (3.6):

(a′) (a) induces isomorphisms $\mathrm{Ext}^{n-i}_c(F, \omega \otimes M) \to \mathrm{Hom\ cont}_{O(Y)}(H^i(F), M)$ for all i.

(b′) (b) induces an isomorphism $\mathrm{Ext}^n(F, \omega \otimes M) \to \mathrm{Hom\ cont}_{O(Y)}(H^0(F), M)$ for all F.

For $X \to Y$ smooth and proper we will show (5.1):

$$\mathrm{Ext}^{n-i}(F, \omega \otimes M) \to \mathrm{Hom}_{O(Y)}(H^i(F), M)$$

is an isomorphism for $i = n$. Under the condition $\mathrm{Ext}^j_{O(Y)}(H^i(F), M) = 0$ for all i and all $j \neq 0$, the map above is an isomorphism for all i, $0 \le i \le n$.

2.4. *The case* $X = \mathbb{A}^n_Y = \mathbb{A}^n_K \times Y$

The points of $\mathbb{A}^n_Y$ are given in coordinates $(z_1, \ldots, z_n, y)$. The sheaf ω is isomorphic to $O_X\, dz_1 \wedge \cdots \wedge dz_n$. For $R \in |K^*|$ we consider $B(R) = \{(z_1, \ldots, z_n, y) \mid \text{all } |z_i| \le R\}$. This is a polydisk over Y. The $\{B(R)\}$ from a cofinal subset for c. From (1.3) and the well known fact that $H^i(X, F) = 0$ for $i \neq 0$ and F coherent on X, it follows that

$$H^1_{B(R)}(\mathbb{A}^1_Y, \omega) = \text{the cokernel of } H^0(\mathbb{A}^1_Y, \omega) \to H^0(\mathbb{A}^1_Y - B(R), \omega)$$
$$H^n_{B(R)}(\mathbb{A}^n_Y, \omega) = H^{n-1}(X - B(R), \omega) \quad \text{for } n \ge 2.$$

In the sequel we will take for convenience $n \ge 2$. The space $X - B(R)$ has the following admissible covering by open subspaces with trivial cohomology for (quasi-)coherent sheaves:

$$\{\{(z_1, \ldots, z_n, y) \in X \mid |z_i| > R\} \mid i = 1, \ldots, n\}.$$

Using Čech-cohomology with respect to this covering, one finds that

$H^i_{B(R)}(\mathbb{A}^n_Y, \omega) = 0$ for $i \neq n$ and that every element $\xi \in H^n_{B(R)}(X, \omega)$ can uniquely be written as infinite sum, converging on $\{(z_1, \ldots, y) \in X \mid \text{all } |z_i| > R\}$, namely

$$\xi \text{ "="} \sum_{\text{all } m_i \leq 0} a_{\boldsymbol{m}} z^{\boldsymbol{m}} \frac{dz}{z}$$

where $\boldsymbol{m} = (m_1, \ldots, m_n)$; $z^{\boldsymbol{m}} = z_1^{m_1} \cdots z_n^{m_n}$; $dz/z = dz_1/z_1 \wedge \cdots \wedge dz_n/z_n$; $a_{\boldsymbol{m}} \in O(Y)$. For $H^n_c(X, \omega) = \varinjlim_R H^n_{B(R)}(X, \omega)$ we have a similar unique expression for the elements as infinite sums. The infinite sum converges on $\{(z_1, \ldots, y) \in X \mid \text{all } |z_i| > R'\}$ for some $R' \in |K^*|$. The *residue map* $\mathrm{Res}_{Y,n} = \mathrm{Res}_n$ given by $\mathrm{Res}_n(\xi) = a_0 \in O(Y)$.

This map is continuous.

Let M be a finitely generated $O(Y)$-module and let I be any subset of $1, \ldots, n$. Put $X_I = \{(z_1, \ldots, z_n, y) \in \mathbb{A}^n_Y \mid |z_i| > R \text{ for all } i \in I\}$. One easily shows that $H^0(X_I, \omega \otimes M) = H^0(X_I, \omega) \otimes M$. It follows that $H^i(X - B(R), \omega \otimes M) = 0$ for $i \neq n-1$ and $H^{n-1}(X - B(R), \omega \otimes M) = H^{n-1}(X - B(R), \omega) \otimes M$. Hence $H^n_c(X, \omega \otimes M) = H^n_c(X, \omega) \otimes M$ and that the trace map $H^n_c(X, \omega \otimes M) \to M$, defined as $\mathrm{Res}_X \otimes 1_M$ is continuous.

2.5. LEMMA. **(The duality for $F = O_X$).** *Let M be any finitely generated $O(Y)$-module. Then*:

1. $\mathrm{Ext}^p_c(O_X, \omega \otimes M) = 0$ *for* $p \neq n$ *and* $\mathrm{Ext}^p(O_X, \omega \otimes M) = 0$ *for* $p \neq 0$.
2. $\mathrm{Ext}^n_c(O_X, \omega \otimes M) = H^n_c(\omega \otimes M) \cong \mathrm{Hom\,cont}_{O(Y)}(H^0(O_X), M)$.
3. $\mathrm{Ext}^0(O_X, \omega \otimes M) = H^0(\omega \otimes M) \cong \mathrm{Hom\,cont}_{O(Y)}(H^n_c(O_X), M)$.

PROOF. The first statement is already proved in (2.4). For the second statement we observe that any element $\xi = \sum a_{\boldsymbol{m}} z^{-\boldsymbol{m}}(dz/z) \in H^n_c(\omega \otimes M)$ induces the $O(Y)$-linear and continuous map

$$f = \sum c_{\boldsymbol{m}} z^{\boldsymbol{m}} \in H^0(O_X) \mapsto \mathrm{Tr}_{X,M}(f\xi) = \sum a_{\boldsymbol{m}} c_{\boldsymbol{m}} \in M.$$

Every continuous $O(Y)$-linear $H^0(O_X) \to M$ has this form for a unique ξ. For the third statement we see that the $O(Y)$-linear map associated with $c = \sum c_{\boldsymbol{m}} z^{\boldsymbol{m}} dz \in H^0(\omega \otimes M)$ is given by

$$l_c : f = \sum f_{\boldsymbol{m}} z^{-\boldsymbol{m}} \frac{dz}{z} \in H^n_c(O_X) \mapsto \mathrm{Tr}_{X,M}(fc) = \sum f_{\boldsymbol{m}} c_{\boldsymbol{m}} \in M.$$

The restriction of l to the Fréchet space $H^n_{B(R)}(O_X) = H^{n-1}(X \setminus B(R), O_X)$ is continuous. Further one easily sees that any continuous $O(Y)$-linear $H^n_c(O_X) \to M$ is equal to l_c for a unique $c \in H^0(\omega \otimes M)$. This finishes the proof.

2.6. PROPOSITION. **Duality for $X = \mathbb{A}^n_Y$.** *Let F be any coherent sheaf on X and let M be a finitely generated $O(Y)$-module.*

1. $\mathrm{Ext}^p_c(F, \omega \otimes M) = 0$ *for* $p < n$.
2. $H^i(F) = 0$ *for* $i \neq 0$.
3. $\mathrm{Ext}^n_c(F, \omega \otimes M) \xrightarrow{\sim} \mathrm{Hom\,cont}_{O(Y)}(H^0(F), M)$.
4. $\mathrm{Hom}(F, \omega \otimes M) \cong \mathrm{Hom\,cont}_{O(Y)}(H^n_c(F), M)$.

PROOF. The second statement is valid because X is a quasi-Stein space (cf. [K2]). The validity of (2.6) for O_X^m follows from (2.5). Let F be a coherent sheaf on X. One would like to prove (2.6) by using a resolution of F by sheaves of the type O_X^m. In general, F has only locally such a resolution. We will use this to prove (2.6).

Fix $R \in \sqrt{|K^*|}$. Let $e_1, \ldots, e_{m_0} \in H^0(F)$ generate $H^0(B(R), F)$ over $O(B(R))$. Define $F_0 = O_X^{m_0} \to F$ by $(0, \ldots, 1, 0, \ldots, 0) \mapsto e_i$ $(1 \le i \le m_0)$. Let F^1 denote the kernel of this map and $\tilde{F}$ the image of this map.

Take $R' \in \sqrt{|K^*|}$ with $R' < R$. Then the exact sequence $0 \to F^1 \to F_0 \to \tilde{F} \to 0$, combined with $\mathrm{Ext}^p_{B(R')}(\tilde{F}, \omega \otimes M) = \mathrm{Ext}^p_{B(R')}(F, \omega \otimes M)$, shows by induction on p that $\mathrm{Ext}^p_{B(R')}(F, \omega \otimes M) = 0$ for $p < n$. This proves statement (1) of (2.6).

For statement (3) we make the following refinement:

(3*) $\mathrm{Ext}^n_{c(R)}(F, \omega \otimes M)) \cong \mathrm{Hom\,cont}_{O(Y)}(F(B(R)), M)$ where $c(R)$ is the family of supports $\{B(R') \mid R' < R \text{ and } R' \in \sqrt{|K^*|}\}$. Further

$$B(R^-) := \{(x_1, \ldots, y) \mid \text{all } |x_i| < R\}$$

and $F(B(R^-))$ is a Fréchet space. We note that $\mathrm{Hom\,cont}_{O(Y)}(F(B(R^-)), M)$ equals $\{l \in \mathrm{Hom}_{O(Y)}(H^0(F), M) \mid$ there exist C and $R' < R$ with $\|l(f)\| \le C \|f\|_{B(R')}$ for all $f \in H^0(F)\}$. Taking $\varinjlim$ on both sides of (3*) one obtains (3).

An inspection of the proof of (2.5) yields the validity of (3*) for O_X and O_X^m. Using the exact sequence of coherent sheaves

$$0 \to F^1 \to F_0 \to \tilde{F} \to 0$$

one finds $\tilde{F}(B(R^-)) = F(B(R^-))$, the exactness of the sequence of Fréchet spaces

$$0 \to F^1(B(R^-)) \to F_0(B(R^-)) \to \tilde{F}(B(R^-)) \to 0$$

and that $\mathrm{Ext}^n_{c(R)}(F, \omega \otimes M) \to \mathrm{Hom}_{O(Y)}(H^0(F), M)$ factors canonically over $\mathrm{Ext}^n_{c(R)}(F, \omega \otimes M) \to \mathrm{Hom\,cont}_{O(Y)}(F(B(R^-)), M)$.

This defines the map in (3*), depending functorially on F. Define $F_1 = O_X^{m_1} \to F^1$ such that $F_1(B(R)) \to F^1(B(R))$ is surjective. The restiction of the complex of sheaves $F_1 \to F_0 \to F \to 0$ to $B(R)$ is exact. In the commutative diagram:

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\mathrm{Ext}^n_{c(R)}(F, \omega \otimes M) \arrow[r, "\alpha"] \arrow[d] & \mathrm{Hom\,cont}_{O(Y)}(F(B(R^-)), M) \arrow[d] \\
\mathrm{Ext}^n_{c(R)}(F_0, \omega \otimes M) \arrow[r, "\alpha_0"] \arrow[d] & \mathrm{Hom\,cont}_{O(Y)}(F_0(B(R^-)), M) \arrow[d] \\
\mathrm{Ext}^n_{c(R)}(F_1, \omega \otimes M) \arrow[r, "\alpha_1"] & \mathrm{Hom\,cont}_{O(Y)}(F_1(B(R^-)), M)
\end{tikzcd}$$

the two columns are exact. The first one since $\mathrm{Ext}^p_{c(R)}(-, \omega \otimes M) = 0$ for $p < n$ and the second one is exact since $F_1(B(R^-)) \to F_0(B(R^-)) \to F(B(R^-)) \to 0$ is an exact sequence of Fréchet spaces. Since α_0 and α_1 are isomorphisms so is α. This proves (3*) and consequently (3).

In order to prove (4) we consider the following statement:

(4*) $\alpha_F : \varinjlim \mathbf{Hom}(F, \omega \otimes M)(B(R')) \to \mathrm{Hom\ cont}_{O(Y)}(H^n_{B(R)}(F), M)$ is an isomorphism.

For $F = O_X$ or O_X^m the existence of this morphism follows from the explicit description of the map in (2.5). For general coherent F on X we consider, as before, the complex $F_1 \to F_0 \to F \to 0$. The exactness of the sequence of Fréchet spaces

$$H^n_{B(R)}(F_1) \to H^n_{B(R)}(F_0) \to H^n_{B(R)}(F) \to 0$$

and the isomorphisms $\alpha_{F_0}, \alpha_{F_1}$ imply the existence of the isomorphism α_F depending functorially on F. This proves (4*) and by taking the projective limit one finds (4).

3. INVARIANCE PROPERTIES OF RES

We start this section by proving that $\mathrm{Res}_n : H^n_c(\mathbb{A}^n_Y, \omega) \to O(Y)$ is invariant under certain automorphisms of $\mathbb{A}^n_Y/Y$. From (3.5) it will follow that Res_n is in fact invariant under *all* automorphisms of $\mathbb{A}^n_Y/Y$.

3.1. LEMMA. *There exists an isomorphism of $O(Y)$-modules $H^n(\mathbb{P}^n_Y, \omega) \cong O(Y)$ such that* $\mathrm{Res}_n = H^n_c(\mathbb{A}^n_Y, \omega) \xrightarrow{\varphi} H^n_c(\mathbb{P}^n_Y, \omega) = H^n(\mathbb{P}^n_Y, \omega) \to O(Y)$. *Further* Res_n *and the trace map are invariant under all affine automorphisms of $\mathbb{A}^n_Y$ over Y.*

PROOF. Since $\mathbb{P}^n_Y$ is proper over Y one has $H^n_c(\mathbb{P}^n_Y, \omega) = H^n(\mathbb{P}^n_Y, \omega)$. By Gaga $H^n(\mathbb{P}^n_Y, \omega)$ is equal to the same cohomology group in the algebraic case. So $H^n(\mathbb{P}^n_Y, \omega) \cong O(Y)$. In general it is difficult to see how Res_n behaves under automorphisms of $\mathbb{A}^n_Y$ over Y. However a σ of the form $\sigma(z_1, \dots, z_n, y) = (\lambda_1 z_1, \dots, \lambda_n z_n, y)$ with all $\lambda_i \in K$ and all $|\lambda_i| = 1$ leaves the $B(R)$'s and the coverings of $X - B(R)$ invariant. Hence σ works on $H^n_c(X, \omega)$ in the obvious way. The extension of σ to an automorphism of $\mathbb{P}^n_Y$ acts trivially on $H^n(\mathbb{P}^n_Y, \omega)$, since $P\mathrm{Gl}(n+1, O(Y))$ acts trivially on $H^n(\mathbb{P}^n_Y, \omega)$. For $\xi \in H^n_c(\mathbb{A}^n_Y, \omega)$ represented by $\sum a_m z^m (dz/z)$ one has $\varphi(\sigma\xi - \xi) = 0$ and $\sigma\xi - \xi = \sum a_m(\lambda^m - 1)(dz/z)$. For a suitable choice of $\lambda_1, \dots, \lambda_n$, i.e. $\lambda^m - 1$ not too small for $\boldsymbol{m} \neq \boldsymbol{0}$, one finds the following consequence: $\ker \varphi = \ker \mathrm{Res}_n$.

This proves the first statement. The second statement follows from the trivial action of $P\mathrm{Gl}(n+1, O(Y))$ on $H^n(\mathbb{P}^n_Y, \omega)$.

3.2. LEMMA. *Let σ denote a Y-automorphism of $\mathbb{A}^n_Y$. There exists a unique $f \in O(Y)^*$ such that* $\mathrm{Res}_n \circ \sigma(\xi) = f.\ \mathrm{Res}_n(\xi)$ *for all* $\xi \in H^n_c(\mathbb{A}^n_Y, \omega)$.

PROOF. $\xi \mapsto \mathrm{Res}_n \circ \sigma(\xi)$ is a continuous $O(Y)$-linear map of $H^n_c(\omega)$ to $O(Y)$.

Apply now (2.5.3) with O_X replaced by ω and $M = O(Y)$. Then $\mathrm{Res}_n \circ \sigma(\xi) = \mathrm{Res}_n(f\xi)$ holds for a unique element $f \in H^0(O_X)$. Using σ^{-1} instead of σ one finds that $f \in H^0(O_X)$ is invertible. The invertible elements of $H^0(O_X)$ are precisely $O(Y)^*$. This proves the statement.

3.3. LEMMA. *Let σ denote a Y-automorphism of $\mathbb{A}^n_Y$ of the form $\sigma(z_1, \dots, z_n, y) = (z_1, \dots, z_m, z_{m+1} + k_{m+1}, \dots, z_n + k_n, y)$ where $k_{m+1}, \dots, k_n$ are holomorphic functions depending on $z_1, \dots, z_m, y$. Then* $\mathrm{Res}_n \circ \sigma = \mathrm{Res}_n$.

PROOF. The σ above is denoted by $[k_{m+1}, \dots, k_n]$. Let $\tilde{\pi}$ be automorphism with $\tilde{\pi}(z_1, \dots, z_m, y) = (\pi z_1, \dots, \pi z_m, z_{m+1}, \dots, z_n, y)$. Let τ be $[l_{m+1}, \dots, l_n]$. Then one easily calculates that $\tilde{\pi}^{-1}\tau_{-1}\tilde{\pi}\tau$

$$= [l_{m+1}(\pi z_1, \dots, \pi z_m) - l_{m+1}(z_1, \dots, z_m), \dots, l_n(\pi z_1, \dots, \pi z_m) - l(z_1, \dots, z_m)].$$

This implies that σ is the composition of $\tilde{\pi}^{-1}\tau^{-1}\tilde{\pi}\tau$ (for suitable τ) and a translation $[a_{m+1}, \dots, a_n]$ with all a_i functions of y. The last transformation leaves Res_n invariant because of (2.7) and $\tilde{\pi}^{-1}\tau^{-1}\tilde{\pi}\tau$ leaves Res_n invariant because of (3.2). This proves the lemma.

3.4. *Definition of Res and Trace of Stein spaces*

Let X/Y denote a relative smooth Stein space of relative dimension n and let $\varphi: X \hookrightarrow \mathbb{A}^N_Y$ denote a closed immersion. Then $\omega_{X/Y}$ is canonical isomorphic to $\mathbf{Ext}^{N-n}(\varphi_* O_X, \omega_N)$ where $\omega_N = \omega_{\mathbb{A}^n_Y/Y}$. The residue map $\mathrm{Res}_{X,\varphi}$ induced by φ is the following:

$$H^n_c(X, \omega_{X/Y}) \cong H^n_c(\mathbb{A}^N_Y, \mathbf{Ext}^{N-n}(\varphi_* O_X, \omega_N)) \xrightarrow[\alpha]{\sim} \mathrm{Ext}^N_c(\varphi_* O_X, \omega_N)$$
$$\xrightarrow[\beta]{\sim} \mathrm{Hom\,cont}_{O(Y)}(H^0(O_X), O(Y)) \xrightarrow[\gamma]{} O(Y).$$

Here α comes from the spectral sequence for Ext^N_c. The spectral sequence has only $H^n_c(\mathbb{A}^N_Y, \mathbf{Ext}^{N-n}(\varphi_* O_X, \omega_N))$ as possible non-zero term. Further β is the map derived from (2.6.3). Finally γ is given by $\gamma(l) = l(1)$ for $l \in \mathrm{Hom\,cont}_{O(Y)}(H^0(O_Y), O(Y))$ where 1 denotes the constant function 1 on X.

It is clear that the resulting map $\mathrm{Res}_{X,\varphi}$ is $O(Y)$-linear and continuous. For any finitely generated $O(Y)$-module M one can consider the same sequence of maps with $\omega_{X/Y}, \omega_N, O(Y)$ replaced by $\omega_{X/Y} \otimes M, \omega_N \otimes M, M$. The corresponding trace map is clearly equal to $\mathrm{tr}_{X,\varphi,M} = \mathrm{Res}_{X,\varphi} \otimes \mathrm{id}_M : H^n_c(X, \omega_{X/Y} \otimes M) \to O(Y) \otimes M = M$.

3.5. THEOREM. *For a relative smooth Stein space X/Y, Residue and Trace do not depend on the chosen closed immersion $X \hookrightarrow \mathbb{A}^N_Y$.*

PROOF. Let X_i $(i = 1, 2, 3)$ denote relative smooth Stein spaces over Y of dimensions n_i. Let $\varphi: X_1 \hookrightarrow X_2$, $\Psi: X_2 \hookrightarrow X_3$, denote closed immersions over Y. Then φ (and similar Ψ) induces a canonical $O(Y)$-linear continuous map

$H_c^{n_1}(X_1, \omega_{X_1/Y}) \xrightarrow{\tilde{\varphi}} H_c^{n_2}(X_2, \omega_{X_2/Y})$. The map $\tilde{\varphi}$ is given by the sequence of maps:

$$\begin{aligned} H_c^{n_1}(X_1, \omega_{X_1/Y}) &\cong H_c^{n_1}(X_2, \mathbf{Ext}^{n_2-n_1}(\varphi_* O_{X_1}, \omega_{X_2/Y})) \\ &\to \mathrm{Ext}_c^{n_2}(\varphi_* O_{X_1}, \omega_{X_2/Y}) \to H_c^{n_2}(X_2, \omega_{X_2/Y}). \end{aligned}$$

The last map is derived from $O_{X_2} \to \phi_* O_{X_1}$ and $\mathrm{Ext}_c^{n_2}(O_{X_2}, \omega_{X_2/Y}) = H^{n_2}(X_2, \omega_{X_2/Y})$. We note that $(\psi\phi)^{\sim} = \tilde{\psi}\tilde{\phi}$.

We consider (as in [Ch]) two closed immersions $\varphi_i : X \hookrightarrow \mathbb{A}_Y^{n_i}$ (of a relative smooth Stein space X/Y of relative dimension n). Then one makes the following commutative diagram:

$$\begin{tikzcd}
 & \mathbb{A}_Y^{n_1} \arrow[rr, "l_1"] \arrow[dr, "k_1"] & & \mathbb{A}_Y^{n_1+n_2} \arrow[dl, "\tau_1"'] \\
X \arrow[ur, "\varphi_1"] \arrow[rr, "(\varphi_1, \varphi_2)"'] \arrow[dr, "\varphi_2"'] & & \mathbb{A}_Y^{n_1+n_2} & \\
 & \mathbb{A}_Y^{n_2} \arrow[rr, "l_2"'] \arrow[ur, "k_2"] & & \mathbb{A}_Y^{n_1+n_2}. \arrow[ul, "\tau_2"']
\end{tikzcd}$$

The morphism $k_1 : \mathbb{A}_Y^{n_1} \to \mathbb{A}_Y^{n_1+n_2} = \mathbb{A}_Y^{n_1} \times_Y \mathbb{A}_Y^{n_2}$ is given by $k_1(a) = (a, \tilde{k}_1(a))$ where $\tilde{k}_1 : \mathbb{A}_Y^{n_1} \to \mathbb{A}_Y^{n_2}$ is a holomorphic map over Y extending $\varphi_2 \circ \varphi_1^{-1} : \varphi_1(X) \to \mathbb{A}_Y^{n_2}$.

Then l_1 is defined by $l_1(a) = (a, 0)$ and τ_1 is given by $\tau_1(a, b) = (a, b + \tilde{k}_1(a))$. Clearly τ_1 is an automorphism of $\mathbb{A}_Y^{n_1+n_2}$ of the form considered in (3.3). The definitions of k_2, l_2, τ_2 are similar. For the very simple map $l_1 : \mathbb{A}_Y^{n_1} \to \mathbb{A}_Y^{n_1} \times \mathbb{A}_Y^{n_2}$, $l_1(a) = (a, 0)$, one can easily verify that

$$H_c^{n_1}(\mathbb{A}_Y^{n_1}, \omega_{\mathbb{A}_Y^{n_1}/Y}) \xrightarrow{\tilde{l}_1} H_c^{n_1+n_2}(\mathbb{A}_Y^{n_1+n_2}, \omega_{\mathbb{A}_Y^{n_1+n_2}/Y}) \xrightarrow{\mathrm{Res}_{n_1+n_2}} O(Y)$$

coincides with Res_{n_1}. Using (3.3) one finds that $\mathrm{Res}_{X,\varphi_1} = \mathrm{Res}_{X,(\varphi_1,\varphi_2)}$. From this and the remark that $\mathrm{Trace}_{X,\varphi,M} = \mathrm{Res}_{X,\varphi} \otimes \mathrm{id}_M$ the theorem follows.

3.6. PROPOSITION. **Duality for Stein spaces.** *Assume that $X \to Y$ is a relative smooth Stein space of relative dimension n. Then the statements of* (2.6) *hold with $\omega = \omega_{X/Y}$.*

PROOF. Let $\varphi : X \to \mathbb{A}_Y^N$ be a closed immersion. One can translate cohomology on X in terms of $\mathbb{A}_Y^N$. Indeed,

$$\omega_{X/Y} \cong \mathbf{Ext}_{O_{\mathbb{A}_Y^N}}^{N-n}(\varphi_* O_X, \omega_{\mathbb{A}_Y^N/Y})$$

$$\varphi_* \mathbf{Ext}_{O_X}^p(F, \omega_{X/Y}) \cong \mathbf{Ext}_{O_{\mathbb{A}_Y^N}}^{p+N-n}(\varphi_* F, \omega_{\mathbb{A}_Y^N/Y})$$

$$\operatorname{Ext}^p_{O_X}(F,\omega_{X/Y}) \cong \operatorname{Ext}^{p+N-n}_{O_{\mathbb{A}^N_Y}}(\varphi_* F, \omega_{\mathbb{A}^N_Y/Y})$$
$$H^\bullet_c(X,F) = H^\bullet_c(\mathbb{A}^N_Y, \varphi_* F) \quad \text{etc.}$$

For more details see [Ch] pp. 165–167.

Then (3.6) is a consequence of (2.6).

3.7. THEOREM. *Let $\varphi: X_1 \to X_2$ be an open immersion of relative smooth Stein spaces of relative dimension n over Y. Then*

$$H^n_c(X_1, \omega_{X_1/Y}) \xrightarrow{\tilde{\varphi}} H^n_c(X_2, \omega_{X_2/Y}) \xrightarrow{\operatorname{Res}_{X_2}} OY)$$

coincides with Res_{X_1}.

PROOF. From (3.6) applied to X_1 we find a unique element $a(X_1,X_2) \in H^0(X_1, O_{X_1})$ such that

$$\operatorname{Res}_{X_2} \circ H^n_c(\varphi)\xi = \operatorname{Res}_{X_1}(a(X_1,X_2)\xi)$$

for all $\xi \in H^n_c(X_1, \omega_{X_1/Y})$.

Let $y_0 \in Y$ be a (ordinary) point. The open immersion of the fibres $X_{1,y_0} \to X_{2,y_0}$ gives rise to some $a(X_{1,y_0}, X_{2,y_0}) \in H^0(X_{1,y_0}, O_{X,y_0})$. One easily sees that $a(X_{1,y_0}, X_{2,y_0})$ is the restriction to X_{1,y_0} of $a(X_1,X_2)$.

Hence it suffices to show (3.7) in case Y is one point.

The first case that we consider is

$$X_1 = \{(z_1,\dots,z_n) \in \mathbb{A}^n_K \mid \text{all } |z_i| < R\} \subset_\varphi X_2 = \mathbb{A}^n_K.$$

As in (2.4) the group $H^n_c(X_1, \omega_{X_1})$ can be described explicitly. The map $\operatorname{Res}_{X_2} \circ H^n_c(\varphi)$ is again explicit. However Res_{X_1} has (a priori) no explicit formula since we have no explicit closed immersion of X_1 into some $\mathbb{A}^N$.

For $u_1,\dots,u_n \in K$, all $|u_i| = 1$, we consider the automorphisms σ of X_1 and X_2 defined by

$$\sigma(z_1,\dots,z_n) = (u_1 z_1, \dots, u_n z_n).$$

The invariance of Res_{X_1} and Res_{X_2} under σ easily implies that $a(X_1,X_2) \in K$. This constant is non-zero since $\operatorname{Res}_{X_2} \circ H^n(\varphi)(dz_1/z_1) \wedge \cdots \wedge (dz_n/z_n) = 1$.

Consider the commutative diagram

$$\begin{array}{ccc}
X_1 & \longrightarrow & X_2 = \mathbb{A}^n_K \\
\uparrow & & \uparrow \\
\{(z_1,\dots,z_n) \in X_1 \mid z_2 = \cdots = z_n = 0\} & \longrightarrow & \{(z_1,\dots,z_n) \in \mathbb{A}^n_K \mid z_2 = \cdots = z_n = 0\} \\
\| & & \| \\
\{z \in \mathbb{A}^1_K \mid |z| < R\} & & \mathbb{A}^1_K
\end{array}$$

The two vertical maps are closed immersions and it follows that $a(X_1,X_2) =$

$a(\{z \in \mathbb{A}_K^1 \mid |z| < R\}, \mathbb{A}_K^1)$. So we are reduced to the case $n=1$. Let us write $a = a(X_1, X_2)$.

Consider again a commutative diagram where all vertical arrows are closed immersions:

$$\begin{tikzcd}
\{(z_1,z_2)\in \mathbb{A}_K^2 \mid |z_1|<R, |z_2|<R\} \arrow[rr] & & \{z\in \mathbb{A}_K^2 \mid |z_1|<R\} \arrow[r] & \mathbb{A}_K^2 \\
\{(0,z_2)\in \mathbb{A}_K^2 \mid |z_2|<R\} \arrow[u] \arrow[r] & \{0\}\times \mathbb{A}_K^1 \arrow[ur] & \{(z_1,0)\in \mathbb{A}_K^2 \mid |z_1|<R\} \arrow[u] \arrow[r] & \mathbb{A}_K^1 \times \{0\} \arrow[u]
\end{tikzcd}$$

This diagram proves $a^2 = a$. Since $a \neq 0$ it follows that $a = 1$. Now we are ready to prove (3.7) for any open immersion $\varphi: X_1 \to X_2$ of smooth Steinspaces of dimension n. Choose a point $p \in X_1$. We want to show $a(X_1, X_2)(p) = 1$. For this we choose two closed immersions $\varphi_i : X_i \hookrightarrow \mathbb{A}_K^{n_i}$ $(i = 1, 2)$ with $\varphi_1(p) = 0$ and $\varphi_2(p) = 0$. After linear base changes in $\mathbb{A}_K^{n_1}$ and $\mathbb{A}_K^{n_2}$ one can find small open polydisks $B_i \subset \mathbb{A}_K^{n_i}$ $(i = 1, 2)$ such that $p \in U = \varphi_1^{-1}(B_1) = \varphi_2^{-1}(B_2)$. The commutative diagram

$$\begin{tikzcd}
U \arrow[r, hook] \arrow[d, "\tilde{\varphi}_1"'] & X_1 \arrow[d, "\varphi_1"] \\
B_1 \arrow[r, hook] & \mathbb{A}^{n_1}
\end{tikzcd}$$

with vertical maps which are closed immersions, proves $a(U, X_1) = a(B_1, \mathbb{A}_K^{n_1}) \equiv 1$. Similarly one shows that $a(U, X_2) \equiv 1$. The evident equality $a(U, X_2) = a(U, X_1)a(X_1, X_2)$ (as functions on U) shows that the restriction of $a(X_1, X_2)$ to U is identical 1. This finishes the proof of the theorem.

4. LOCALIZING THE COHOMOLOGY

4.1. We consider the following situation: Y an affinoid space, $X \to Y$ a morphism of analytic spaces and X separated and quasi-compact. Let $U \subset X$ be an admissible open subset which is a Stein space over Y. Further F denotes any abelian sheaf on X.

THEOREM. *There exists a subsheaf F_U of F with the following properties:*

1. $(F_U)_p \cong F_p$ *for every geometric point p containing U.*
2. $(F_U)_p = 0$ *for every point p with $U \notin p$.*
3. $H^i(X, F_U) = H_c^i(U, F)$ *where $c = c(U/Y)$ and all i.*
4. $F \mapsto F_U$ *is a functor.*

4.2. PROOF. U has an admissible covering $\bigcup_{n=1}^{\infty} U_n$ with all U_n affinoid and $U_{n-1} \Subset_Y U_n$ for $n \geq 2$. Let $t_1, \ldots, t_n$ denote topological generators of $O(U_n)$ over $O(Y)$ such that $U_{n-1} \subset \{u \in U_n \mid \text{all } |t_i(u)| < \varrho\}$ for some $\varrho < 1$. Define now $\tilde{U}_n = \{u \in U_n \mid \text{all } |t_i(u)| < \varrho\}$. Then $\tilde{U}_n/Y$ is again a Stein space. Suppose that we have proven the theorem for each $\tilde{U}_n$. Then one easily sees that $F_U := \varinjlim F_{\tilde{U}_n}$ has the required properties.

So we may suppose that there exists some affinoid $Z \subset X$, with topological generations $t_1, \ldots, t_m$ of $O(Z)$ over $O(Y)$ such that $U = \{z \in Z \mid \text{all } |t_i(z)| < \varrho\}$ for some $\varrho < 1$. Choose a $\varrho_1 \in \sqrt{|K^*|}$ with $\varrho < \varrho_1 < 1$ and let $Z(\varrho_1)$ denote $\{z \in Z \mid \text{all } |t_i(z)| \le \varrho_1\}$. Suppose that we can prove the theorem for the inclusion $U \subset Z$. Let F be any abelian sheaf on X. Let $F|_Z$ denote the restriction of F to Z. Then we have already a sheaf $(F|_Z)_U$. Define now a subsheaf F_U of F by $F_U|_Z = (F|_Z)_U$ and $F_U|_{X - Z(\varrho_1)} = 0$. This is possible since $\{U, X - Z(\varrho_1)\}$ is an admissible covering of X and $(F|_Z)_U|_{Z - Z(\varrho_1)} = 0$. One easily sees that F_U has the required properties. So we are reduced to prove the theorem in case X is affinoid and $U = \{x \in X \mid \text{all } |t_i(x)| < \varrho\}$ where $\varrho < 1$ and $t_1, \ldots, t_m$ are topological generators of $O(X)$ over $O(Y)$. Let F be any abelian sheaf on X. Define for any affinoid $A \subset X$ the subgroup $G(A)$ of $F(A)$ by $f \in F(A)$ lies in $G(A)$ if and only if there exists an affinoid $B = \{x \in X \mid \text{all } |t_i(x)| \le \varrho'\} \subset U$ with $f|_{(X-B) \cap A} = 0$. This makes sense since $(X - B) \cap A$ is an admissible open set for the strong G-topology on X. Clearly $G(A)$ is a subgroup of $F(A)$. For any inclusion of affinoids $A_1 \subset A_2$ the restriction map $F(A_2) \to F(A_1)$ maps $G(A_2)$ into $G(A_1)$. So $A \mapsto G(A)$ is a pre-sheaf on affinoids. Let $\{A_1, \ldots, A_s\}$ denote a finite covering of some affinoid A by affinoids. The exactness of $0 \to G(A) \to \bigoplus G(A_i) \to \bigoplus G(A_i \cap A_j)$ follows easily from the sheaf property of F. So G is a sheaf for the G-topology on X defined by the affinoid subsets of X. According to [BGR], G extends uniquely to a sheaf on X for the strong G-topology. Let p denote any geometric point [P] of X containing U (i.e. p contains some affinoid $B = \{x \in X \mid \text{all } |t_i(x)| \le \varrho'\} \subset U$). Then clearly $G_p \to F_p$ is an isomorphism, since $G|_B = F|_B$. Suppose now $U \notin p$. For any $\varrho' < \varrho$ we consider the covering of X: $\{x \in X \mid \text{all } |t_i| \le \varrho'\}, \{x \in X \mid |t_1| \ge \varrho'\}, \ldots, \{x \in X \mid |t_m| \ge \varrho'\}$. It follows that there exists an $i \in \{1, \ldots, m\}$ such that $\{x \in X \mid |t_i| \ge \varrho'\} \in \varrho$ for all $\varrho' < \varrho$.

Choose now an affinoid $A \in p$ and some $f \in G(A)$. Then the restriction of f to $\{a \in A \mid |t_i(a)| \ge \varrho_1\}$ is zero for some $\varrho_1 < \varrho$ and $\varrho_1 < \varrho$ and ϱ_1 close enough to ϱ. Now $\{a \in A \mid |t_i(a)| \ge \varrho_1\} = A \cap \{x \in X \mid |t_i(x)| \ge \varrho_1\}$ belongs to p. This shows that the image of f in G_p is zero. Hence $G_p = 0$.

Of course G will be the sheaf F_U of the theorem. We have already proven the properties (1) and (2). Property (4) follows from the definition of G. Further $H^0(X, F_U)$ is already defined as $H^0_c(U, F)$ with $c = c(U/Y)$.

Using an injective resolution $0 \to F \to I_0 \to I_1 \to I_2 \to \cdots$ we find a resolution $0 \to F_U \to (I_0)_U \to (I_1)_U \to \cdots$ of F_U. It can be shown that the sheaves $(I_0)_U$ are also injective. From this statement (3) follows.

4.3. REMARKS. In [P] geometric points are defined only for affinoid spaces. For a quasi-compact space X, i.e. X is a finite union of affinoids $X_1, \ldots, X_r$, a geometric point of X is by definition a geometric point of some X_i. The general theory for abelian sheaves and geometric points for quasi-compact X works as in [P]. For X not quasi-compact the theory of abelian sheaves and geometric points is probably different. In any case one sees that (4.1) can no longer be true by taking the example $X=U=$a Stein space and $F=O_X$.

5. THE DUALITY THEOREM

5.1. **Main Theorem.** *Let $f: X \to Y$ satisfy:*

1. *Y is an affinoid space.*
2. *X is separated.*
3. *$X \to Y$ is smooth and proper of relative dimension n. Let F be a coherent sheaf on X and let M be an $O(Y)$-module. Then*
   (a) *There exists a trace map* $\mathrm{tr}_M : H^n(\omega_{X/Y} \otimes M) \to M$.
   (b) *The canonical maps, induced by* tr_M,

$$\mathrm{Ext}^{n-i}(F, \omega_{X/Y} \otimes M) \to \mathrm{Hom}_{O(Y)}(H^i(F), M) \quad (0 \le i \le n)$$

are isomorphisms if $\mathrm{Ext}^j_{O(Y)}(H^i(F), M) = 0$ *for $j \neq 0$ and all i.*

5.2. REMARKS. The condition on $\mathrm{Ext}^{\bullet}_{O(Y)}$ is clearly satisfied if either M is an injective $O(Y)$-module or all $H^i(F)$ are projective $O(Y)$-modules. According to [K1] the groups $H^i(F)$ are finitely generated $O(Y)$-modules. Our definition of the locally convex topology on $H^i(F)$ is such that $H^i(F)$ has the canonical topology as finitely generated $O(Y)$-module. It follows that $\mathrm{Hom}_{O(Y)}(H^i(F), M) = \mathrm{Hom\,cont}_{O(Y)}(H^i(F), M)$.

5.3. THE PROOF. According to the definition of proper we can write $X = X_1 \cup \cdots \cup X_r = X_1' \cup \cdots \cup X_r'$ where all X_i, X_i' are affinoids such that $X_i \Subset_Y X_i'$ for $i = 1, \dots, r$. Let $t_1, \dots, t_m$ denote topological generators of $O(X_i')$ over $O(Y)$ such that $X_i \subset U_i := \{x \in X_i' \mid \text{all } |t_i(x)| < 1\}$. Then $\{U_1, \dots, U_r\}$ is an admissible covering of X by Stein spaces over Y. All intersections $U_{i_1} \cap \cdots \cap U_{i_t}$ are also Stein spaces over Y.

Let F be any abelian sheaf on X. According to (4.1) we can form an exact sequence

$$\bigoplus_{i<j} F_{U_i \cap U_j} \to \bigoplus F_{U_i} \to F \to 0.$$

Taking cohomology and applying again (4.1) one finds an exact sequence

$$\bigoplus_{i<j} H^n_c(U_i \cap U_j, F) \to \bigoplus_i H^n_c(U_i, F) \to H^n(X, F) \to 0.$$

Substitute $F = \omega_{X/Y}$, take $\mathrm{Hom\,cont}_{O(Y)}(\ , N)$ with N any finitely generated $O(Y)$-module and apply the duality theorem for Stein spaces (3.6) then one obtains the exact sequence

$$\bigoplus_{i<j} H^0(U_i \cap U_j, O_X \otimes N) \xleftarrow{\alpha} \bigoplus_i H^0(U_i, O_X \otimes N) \leftarrow \mathrm{Hom}_{O(Y)}(H^n(X, \omega_{X/Y}), N) \leftarrow 0.$$

The map α is the usual map since we have shown in (3.7) that for the open immersion of Stein spaces $U_i \cap U_j \subset U_i$ the residue map is invariant. This yields a canonical isomorphism $H^n(X, \omega_{X/Y}) \to O(Y)$, again called Res_X. The same reasoning applied to $F = \omega_{X/Y} \otimes M$, where M is a finitely generated $O(Y)$-module, yields an isomorphism $\mathrm{Tr}_M : H^n(X, \omega_{X/Y} \otimes M) \to M$. Let M be any $O(Y)$-module. Since X is a quasi-compact space $H^n(X, \omega_{X/Y} \otimes M)$ equals $\varinjlim \{H^n(X, \omega_{X/Y} \otimes N) \mid N \subset M \text{ and } N \text{ finitely generated}\}$. So for general M one has an isomorphism $\mathrm{Tr}_M : H^n(X, \omega_{X/Y} \otimes M) \to M$. In order to prove (b) we consider the case $i = n$ and the diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\mathrm{Hom}(F, \omega_{X/Y} \otimes M) \arrow[r] \arrow[d] & \mathrm{Hom}_{O(Y)}(H^n(F), M) \arrow[d] \\
\bigoplus \mathbf{Hom}(F, \omega_{X/Y} \otimes M)(U_i) \arrow[r, "\alpha"] \arrow[d] & \mathrm{Hom\,cont}_{O(Y)}(\bigoplus (H_c^n(U_i, F), M) \arrow[d] \\
\bigoplus \mathbf{Hom}(F, \omega_{X/Y} \otimes M)(U_i \cap U_j) \arrow[r, "\beta"] & \mathrm{Hom\,cont}_{O(Y)}(\bigoplus H_c^n(U_i \cap U_j, F), M).
\end{tikzcd}$$

Property (3.7) of Res proves that the diagram is commutative. The two columns are exact. Further (3.6) implies that α and β are isomorphisms. Hence this proves (b) with $i = n$ and any $O(Y)$-module M.

The functorial isomorphism $\mathrm{Hom}(F, \omega_{X/Y} \otimes M) \to \mathrm{Hom}_{O(Y)}(H^n(F), M)$ (for coherent O_X-modules) implies that the derived functors are also isomorphic. The derived functors of the lefthand side are $\mathrm{Ext}^i(F, \omega \otimes M)$. Under the stated condition in the theorem the derived functors of the righthand side are $\mathrm{Hom}_{O(Y)}(H^{n-i}(F), M)$. This proves the theorem.

5.4. REMARK. It is obvious how to generalize (5.1) to the case "$f: X \to Y$ a proper smooth map of rigid analytic spaces". The natural setting would use derived categories along to lines of [H]. A generalization to the case where f is not smooth (but still proper) seems also possible. However this will involve at least the same amount of technicalities as the complex analytic case.

Proper analytic spaces and Stein spaces are two examples of rigid analytic spaces "without boundary" as defined in [L2], p. 369. It is probable that Serre duality is valid for spaces without boundary (cf. [L2], p. 370). The method in this paper can be adapted to prove a Serre duality for $X \to Y$ relatively smooth, separated and such that X is a finite union of Stein spaces over Y.

REFERENCES

[BGR] Bosch, S., U. Güntzer and R. Remmert - Non archimedean Analysis. Grundlehren Math. Wiss. **261**, Springer Verlag, Berlin (1984).

[B-S] Bănică, C. and O. Stănăsilă - Méthodes algébriques dans la théorie globale des espaces complexes. Vol. 2, Gauthier-Villars (1966).

[Ch] Chiarellotto, B. - Duality in Rigid Analysis. Lecture Notes in Math. 1454, p. 142–172 (1990).

[H] Hartshorne, R. - Residues and Duality. Lecture Notes in Math. **20** (1966).

[K1] Kiehl, R. - Der Endlichkeitsatz für eigentliche Abbildungen in der nichtarchimedische Funktionentheorie. Inventiones Math. **2**, p. 191-214 (1967).

[K2] Kiehl, R. - Theorem A and Theorem B in der nichtarchimedischen Funktionentheorie. Inventiones Math. **2**, p. 256-273 (1967).

[L1] Lütkebohmert, W. - Steinsche Räume in der nicht-archimedischen Funktionentheorie. Schriftreihe Math. Inst. Univ. Münster. **2** Serie, Heft 6 (1973).

[L2] Lütkebohmert, W. - Formal algebraic and rigid-analytic geometry. Math. Ann. **286**, p. 341-371 (1990).

[P] Put, M. van der - Cohomology on affinoid spaces. Compositio Math. **45** (2), p. 165–198 (1982).